UNITÉ DE RECHERCHE
INRIA-RENNES

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: (1) 39 63 55 11

Rapports de Recherche

N° 1299

Programme 5
Automatique, Productique,
Traitement du Signal et des Données

MULTISCALE SIGNAL PROCESSING : FROM QMF TO WAVELETS

Albert BENVENISTE

Octobre 1990

RR . 1299

IRISA INSTITUT DE RECHERCHE EN INFORMATIQUE ET SYSTEMES ALEATOIRES

Campus Universitaire de Beaulieu
35042 - RENNES CEDEX
FRANCE
Téléphone : 99.36.20.00
Télex : UNIRISA 950 473F
Télécopie : 99.38.38.32

Multiscale Signal Processing: from QMF to wavelets.

Albert Benveniste
IRISA-INRIA, Campus de Beaulieu
35042 RENNES CEDEX, FRANCE

July 11, 1990

Publication Interne n°550 - Septembre 1990, 28 Pages

Abstract

This is a short course intended to relate to each other the following topics: multirate filtering and maximally decimated QMF filter banks, multiresolution analysis and multiscale signal analysis and associated wavelet transforms.

Traitement du signal multirésolution: des QMF aux ondelettes

Résumé

Nous présentons des notes brèves de cours dont le but est de relier entre elles les notions suivantes: traitement du signal multirésolution et bancs de filtres QMF à décimation maximale d'une part, et analyse multirésolution de signaux et ondelettes associées d'autre part.

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE
(U.R.A. 227)
UNIVERSITE DE RENNES I I.N.S.A. DE RENNES

INSTITUT NATIONAL DE RECHERCHE
EN INFORMATIQUE ET EN AUTOMATIQUE
(UNITE DE RECHERCHE DE RENNES)

Multiscale Signal Processing: from QMF to wavelets.

Albert Benveniste
IRISA-INRIA, Campus de Beaulieu
35042 RENNES CEDEX, FRANCE

September 13, 1990

Abstract

This is a short course intended to relate to each other the following topics: multirate filtering and maximally decimated QMF filter banks, multiresolution analysis and multiscale signal analysis and associated wavelet transforms.

1 Introduction

Multirate filtering [7] is now recognized as an area of increasing importance in digital signal processing. The key issue in this area is how to handle properly the aliasing due to sampling below the Nyquist rate. Maximally decimated filter banks have been introduced [12, 21, 22] that allow to design filter banks with an exact saving of the global sampling rate (eg. a 2–filter bank must involve downsampling by a factor of 2 in each subband).

Multiscale signal analysis [10] appears as an emerging alternative technique to Fourier analysis. Wavelet and related transforms are becoming increasingly popular in this area. *Multiscale signal recognition,* i.e. performing multiscale pattern recognition on signals, is certainly a desirable objective, although no well established approach is available today for this purpose.

Finally, *multiresolution analyses* of L^2–spaces of functions or kernels, and related wavelets [19], recently proved an extremely powerful toolbox for highly demanding mathematical problems in harmonic and functional analysis. They provide a very effective approach to derive new approximations and expansions of functions, or operators, that are classically difficult to handle (integral equations, singular integral, pseudodifferential operators). These theories also provide a new theoretical support to the general area of *multigrid methods* [13, 17].

It has been recently recognized that these apparently different topics are closely related to each other. This is precisely the subject of this short course, mostly intended to readers with signal processing and otherwise general mathematical background. For the sake of simplicity, we decided to concentrate on the essential features, leaving aside unnecessary technicalities: $1D$–domains, 2–filter banks, and infinite signals are only considered here. To emphasize the essentials of the topic, we also tried to separate as much as possible in our presentation the algebraic aspects (mostly related to the theory of *Quadrature Mirror Filters -QMF-* in signal processing) from those relevant to harmonic and functional analysis (issues of convergence and approximation). This is the originality of the paper. Finally, to avoid overlength and tedious calculations, we decided to have a relatively compact presentation of the mathematics, so that this paper is indeed worth of a pencil-and-paper aided reading.

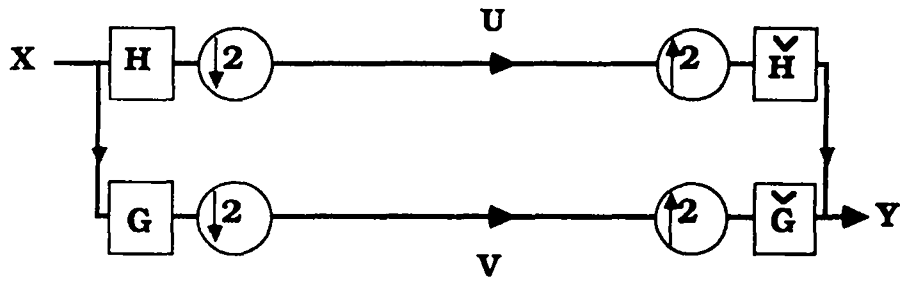
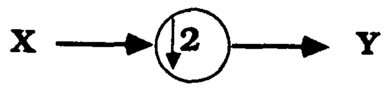


Figure 1: Maximally decimated filter bank

2 QMF Banks and the Polyphase approach

2.1 Down– and up-sampling



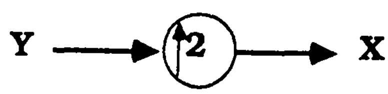
The above diagram will represent down–sampling by a factor of 2 throughout this paper. Consequently,

$$\begin{aligned} X(z) &= \sum_n x_n z^{-n} \\ Y(z^2) &= \sum_n y_n z^{-2n} = \sum_n x_{2n} z^{-2n} \end{aligned}$$

hence

$$Y(z^2) = \frac{1}{2}\left[X(z) + X(-z)\right] \tag{2.1}$$

Similarly



represents up–sampling by a factor of 2, so that

$$X(z) = \sum_n x_n z^{-n} = \sum_n y_n z^{-2n} = Y(z^2) \tag{2.2}$$

2.2 Maximally decimated filter banks

The diagram of the figure 1 depicts such a filter bank. Using formulae (2.1,2.2), the input–output map is written as follows:

$$Y(z) \quad = \quad \check{H}(z)U(z^2) + \check{G}(z)V(z^2)$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\check{H}(z)H(z)+\check{G}(z)G(z)\right]X(z) &\quad (2.3)\\
&+ \frac{1}{2}\left[\check{H}(z)H(-z)+\check{G}(z)G(-z)\right]X(-z) &\quad (2.4)
\end{aligned}
$$

In this formula, the expression (2.4) represents the *aliasing* component, whereas (2.3) represents the linear transfer component. Hence, for this linear map to be an (*aliasfree*) time–invariant filter, we must have

$$\check{H}(z)H(-z)+\check{G}(z)G(-z)=0 \tag{2.5}$$

and, for *perfect reconstruction*

$$\check{H}(z)H(z)+\check{G}(z)G(z)=1 \tag{2.6}$$

up to a delay.

A sketch of history. The first paper to consider this problem is [12], where the following solution is proposed to satisfy the condition (2.5) for antialising:

$$\check{H}(z)=G(-z)\ ,\ \check{G}(z)=-H(-z)$$

Then, the following choice for H

$$G(z)=H(-z)$$

results in the following condition for perfect reconstruction:

$$H^2(z)-H^2(-z)=1$$

for which no pleasant exact solution does exist. The first satisfactory solution was due to Smith & Barnwell in 1983 [21] and is explained next. Select H such that

$$H(z)H(z^{-1})+H(-z)H(-z^{-1})=1$$

(i.e. white noise $\rightarrow \boxed{H} \rightarrow \boxed{\downarrow 2} \rightarrow$ white noise).
Then the following choice

$$
\begin{array}{cccccc}
H(z) & H(z^{-1}) & + & H(-z) & H(-z^{-1}) & = \ 1\\
 & \updownarrow & & \updownarrow & \updownarrow & \\
 & \check{H}(z) & & z^{-1}\check{G}(z) & zG(z) &
\end{array}
$$

satisfies both antialiasing (2.6) and perfect reconstruction (2.5) conditions.

2.3 The polyphase approach

Consider the diagram of the figure 2: It certainly satisfies both antialiasing and perfect reconstruction conditions. On the other hand, this diagram is clearly equivalent to the next one of the figure 3 provided that

$$E(z)\check{E}(z)=I \tag{2.7}$$

This last diagram is now redrawn as in the figure 4 or, equivalently, as in the figure 5 by setting

$$E(z)=\begin{bmatrix} H_0(z) & H_1(z)\\ G_0(z) & G_1(z)\end{bmatrix}\ ,\ \check{E}(z)=\begin{bmatrix} \check{H}_1(z) & \check{G}_1(z)\\ \check{H}_0(z) & \check{G}_0(z)\end{bmatrix} \tag{2.8}$$

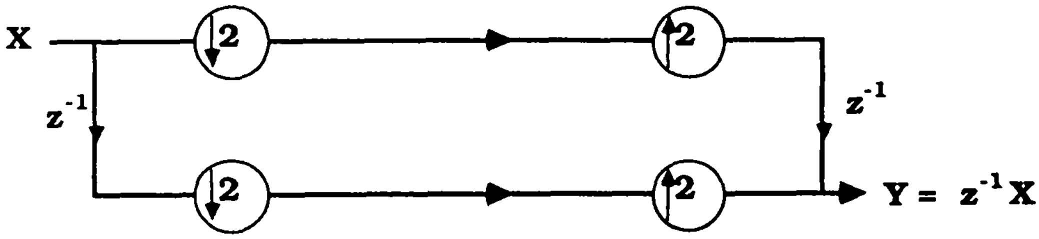


Figure 2: The polyphase approach: stage 1

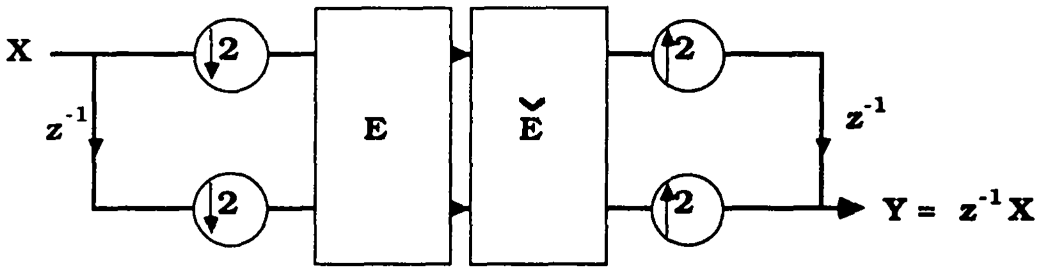


Figure 3: The polyphase approach: stage 2

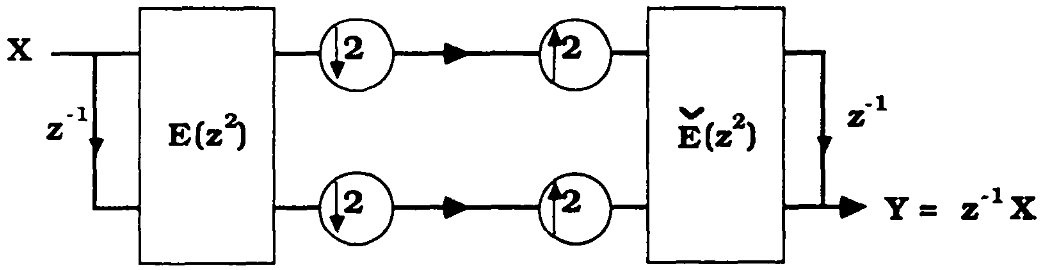


Figure 4: The polyphase approach: stage 3

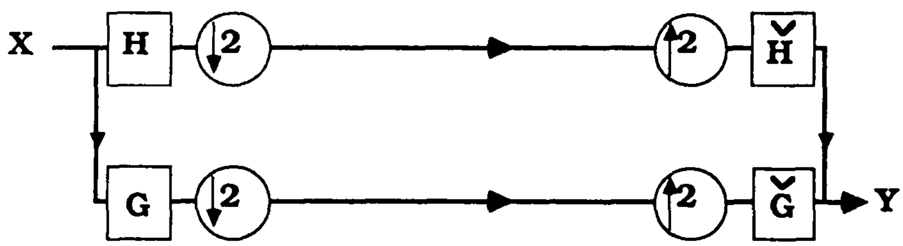


Figure 5: The polyphase approach: stage 4

and

$$\begin{aligned} H(z) &= H_0(z^2) + z^{-1}H_1(z^2) \\ G(z) &= G_0(z^2) + z^{-1}G_1(z^2) \end{aligned} \tag{2.9}$$

and similarly for $\check{H}, \check{G}$. Checking for solutions to (2.9) and then applying formulae (2.8,2.7) is known as the *polyphase approach to the QMF problem.* In the sequel, we shall concentrate on matrix *polynomial* solutions to (2.9) to get FIR maximally decimated filter banks.

3 QMF synthesis

In this section, we check for 2–port transfer functions E and $\check{E}$ satisfying

$$E\check{E} = I \tag{3.10}$$

Several solutions exist. Of particular interest are *polynomial* solutions we shall discuss throughtout this course.

3.1 Non unitary QMF synthesis

This word refers to any solution $(E, \check{E})$ of equation (3.10). Such a solution does exist if and only if the 2–port polynomial matrix E is a unit in the ring of polynomial matrices, i.e. if it is *unimodular,* say[1]

$$1 \equiv \det E(z) = H_0(z)G_1(z) - H_1(z)G_0(z)$$

Then we take

$$\check{E} = E^{-1} = \begin{bmatrix} G_1 & -H_1 \\ -G_0 & H_0 \end{bmatrix}$$

which yields, up to a delay,

$$\begin{aligned} \check{H}(z) &= -G_0(z^2) - z^{-1}G_1(z^2) \\ \check{G}(z) &= H_0(z^2) - z^{-1}H_1(z^2) \end{aligned} \tag{3.11}$$

Finally, given H, the problem reduces to a *Bezout or Diophantine equation* to find G. A solution does exist if and only if the pair (H_0, H_1) is coprime. Cascade parametrizations of unimodular E matrices are provided in [20], which structurally guaranty *linear phase* properties for the analysis and synthesis filter banks.

3.2 Unitary QMF synthesis

The question is now the following: find E *lossless,* i.e. such that

$$E(z)E^T(z^{-1}) = I$$

and take $\check{E}(z) = E^T(z^{-1})$ up to a delay. Now, the following equivalences are easy to check:

$$E = \begin{bmatrix} H_0 & H_1 \\ G_0 & G_1 \end{bmatrix} \quad \text{lossless}$$

[1] up to a delay and a constant gain we don't care here

$$\Updownarrow \text{ for } z = e^{i\omega}$$

$$\begin{aligned}
1 &= |H_0|^2 + |G_0|^2 = |H_1|^2 + |G_1|^2 \\
&= |H_0|^2 + |H_1|^2 = |G_0|^2 + |G_1|^2 && (3.12) \\
0 &= \overline{H}_0 H_1 + \overline{G}_0 G_1 = H_0\overline{G}_0 + H_1\overline{G}_1 && (3.13)
\end{aligned}$$

$$\Updownarrow$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \triangleq J \text{ lossless} \tag{3.14}$$

Now, from (3.13) we deduce the following equalities

$$\frac{G_1(e^{i\omega})}{H_0(e^{-i\omega})} = -\frac{H_1(e^{i\omega})}{G_0(e^{-i\omega})} = -\frac{G_0(e^{i\omega})}{H_1(e^{-i\omega})} = e^{i\phi(\omega)}$$

where the last one is obtained by noticing that $\overline{z} = 1/z \Rightarrow |z| = 1$. As a consequence there must exist an all-pass filter $I(z)(|I(e^{i\omega})|^2 = 1)$ such that

$$\begin{aligned}
G_1(z) &= I(z)H_0(z^{-1}) \\
G_0(z) &= -I(z)H_1(z^{-1})
\end{aligned}$$

whence

$$\begin{aligned}
G(z) &= G_0(z^2) + z^{-1}G_1(z^2) \\
&= z^{-1}I(z^2)\left(H_0(z^{-2}) - zH_1(z^{-2})\right) \\
&= z^{-1}I(z^2)H(-z^{-1})
\end{aligned}$$

If G polynomial is wanted, we must take $I(z)$ = pure delay. Combining this with J lossless yields

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2 \tag{3.15}$$

or, equivalently

$$|H\left(e^{i\omega}\right)|^2 + |H\left(e^{i(\omega+\pi)}\right)|^2 = 2 \tag{3.16}$$

These equivalent conditions will be referred to in the sequel as the *Quadrature Mirror Filter (QMF) condition.*

Summary for unitary QMF synthesis

E lossless

$$\Updownarrow$$

J lossless

$$\Updownarrow$$

$$\boxed{\begin{array}{c}\textbf{QMF: } H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2 \\ G(z) = z^{-1}H(-z^{-1})\end{array}}$$

Finally, take, up to a delay,

$$\check{H}(z) = H(z^{-1}) \ , \ \ \check{G}(z) = G(z^{-1}) \tag{3.17}$$

3.3 Synthesis of 2-port lossless FIR polynomial transfer functions.

Direct cascade parametrization of lossless 2–port transfer functions. The following elementary block

$$E_\alpha(z) \triangleq \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

is lossless. Hence, so is

$$E(z) = \prod_{m=n}^{1} E_{\alpha_m}(z) \tag{3.18}$$

Conversely, we are given

$$E = \begin{bmatrix} H_0 & H_1 \\ G_0 & G_1 \end{bmatrix}$$

lossless. We can assume $\det E \equiv 1$, which implies

$$H_{0,n}G_{1,n} = H_{1,n}G_{0,n} \ (n = \ \deg E) \tag{3.19}$$

($H_{i,m}$ coefficient of z^{-m} in H_i). Consider

$$\begin{bmatrix} \tilde{H}_0 & \tilde{H}_1 \\ \tilde{G}_0 & \tilde{G}_1 \end{bmatrix} \triangleq \begin{bmatrix} \cos\alpha & -\sin\alpha \\ z\sin\alpha & z\cos\alpha \end{bmatrix} \begin{bmatrix} H_0 & H_1 \\ G_0 & G_1 \end{bmatrix}$$

From (3.19) we deduce

$$\exists_1 \alpha = \alpha_n : d^o \tilde{H}_i \leq n-1, i = 0,1 \tag{3.20}$$

On the other hand, with $\alpha = \alpha_n$, and using (3.20), we get

$$\left.\begin{array}{rcl} G_1(z) & = & z^{-n}H_0(z^{-1}) \\ G_0(z) & = & -z^{-n}H_1(z^{-1}) \end{array}\right\} \Rightarrow \tilde{G}_i = z^{-1}\tilde{\tilde{G}}_i, i = 0,1$$

for some $\tilde{\tilde{G}}_i$. Hence, setting $E = E_n$, we get the decomposition

$$E_n = \begin{bmatrix} \cos\alpha_n & \sin\alpha_n \\ -\sin\alpha_n & \cos\alpha_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} E_{n-1},$$

$$\deg E_{n-1} \leq n-1$$

showing that the cascade decomposition (3.18) may be used as a general parametrization of 2–port lossless polynomial transfer functions; such a factorization is originally due to Potapov in the fifties. See also [8] for an overview of wave digital filter synthesis, where most of the classical lossless digital filter design techniques are presented following a circuit theoretic point of view. An alternative related parametrization is

$$E(z) = \prod_{m=n}^{1} \begin{bmatrix} 1 & z^{-1}k_m \\ -k_m & z^{-1} \end{bmatrix}$$

which is equivalent to (3.18) up to a constant normalization gain. This parametrization is used by Vaidyanathan [23] to synthetize a {low–pass, high–pass} QMF pair using optimization techniques.

I. Daubechies' QMF unitary synthesis. {Low-pass, high-pass} QMF pairs can be synthetized in an alternative way. Take $H(z)$ of the form

$$H(z) = (1 + z^{-1})^N \tilde{H}(z) \quad \text{(typically low-pass)}$$

Then the condition $2 = H(z)H(z^{-1}) + H(-z)H(-z^{-1})$ is rewritten as

$$\begin{aligned} 1 &= \left(1 + \frac{z + z^{-1}}{2}\right)^N \tilde{H}(z)\tilde{H}(z^{-1}) \\ &+ \left(1 - \frac{z + z^{-1}}{2}\right)^N \tilde{H}(-z)\tilde{H}(-z^{-1}) \end{aligned}$$

Consider the equation

$$1 = \left(1 + \frac{z + z^{-1}}{2}\right)^N Q(z) + \left(1 - \frac{z + z^{-1}}{2}\right)^N Q(-z) \tag{3.21}$$

where $Q(z^{-1}) = Q(z)$. I. Daubechies [9] proved the following important result: *there exists a unique solution to (3.21) of degree $2N$, and this solution satisfies*

$$Q\left(e^{i\omega}\right) \geq 0$$

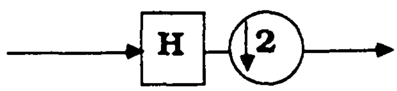
Then QMF synthesis reduces to a polynomial spectral factorization problem.

4 Hilbert space structures of *orthonormal* QMF syntheses.

In this section, we investigate how orthonormal QMF banks give raise to some orthonormal decompositions of l^2–spaces of signals. Such decompositions turn out to be useful (for instance) for coding purposes.

4.1 Basic decompositions

In what follows, we select a given discrete time index set that will serve as a reference for the various sampling rates we shall consider in the sequel. We denote by $\mathbf{Z}$ this discrete time reference. We denote by $l^2(n\mathbf{Z})$ (n integer) the space of square integrable signals x_k such that $x_k \neq 0$ only if k is a multiple of n. Similarly, we denote by $l^2(\frac{1}{n}\mathbf{Z})$ the space of square integrable signals that are *upsampled* at a rate n. Using these notations, the following diagram



induces an operator

$$\mathcal{H} \; : \; l^2(\mathbf{Z}) \longrightarrow l^2(2\mathbf{Z})$$

and similarly for $\mathcal{G}$. Vice-versa

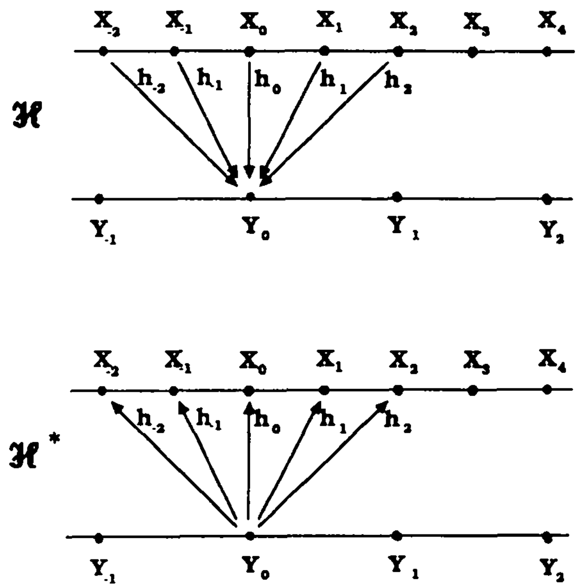
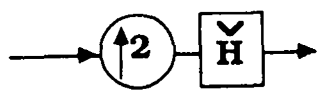


Figure 6: $\mathcal{H}$ and $\mathcal{H}^*$



induces an operator

$$\check{\mathcal{H}} \; : \; l^2(2\mathbf{Z}) \longrightarrow l^2(\mathbf{Z})$$

Now, if we take the case $E = I$ in the diagram of the figure 3 we get immediately the identities

$$\begin{aligned} I &= \check{\mathcal{H}}\mathcal{H} + \check{\mathcal{G}}\mathcal{G} \\ I &= \mathcal{H}\check{\mathcal{H}} = \mathcal{G}\check{\mathcal{G}} \\ 0 &= \mathcal{H}\check{\mathcal{G}} = \mathcal{G}\check{\mathcal{H}} \\ \check{\mathcal{H}} &= \mathcal{H}^* \; , \; \check{\mathcal{G}} = \mathcal{G}^* \end{aligned} \tag{4.22}$$

where $.^*$ denotes the adjoint. Introducing E lossless yields an orthonormal change of basis in $l^2(2\mathbf{Z}) \times l^2(2\mathbf{Z})$, hence the above identities remain valid. Consequently, using the Mason rule, we may draw the $\mathcal{H}$ and $\mathcal{H}^*$ operators as in the figure 6. The formulae (4.22) yield the decomposition

$$\begin{aligned} \mathcal{V}_0 &\triangleq l^2(\mathbf{Z}) = \operatorname{Im}(\mathcal{H}^*) \oplus \operatorname{Im}(\mathcal{G}^*) \\ &\triangleq \mathcal{V}_{-1} \oplus \mathcal{W}_{-1} \end{aligned} \tag{4.23}$$

we shall use in different ways in the sequel.

4.2 A "Gabor"-like coarse–scale decomposition

Using (4.23), the diagram of the figure 7 yields on its leaves a decomposition into signals at

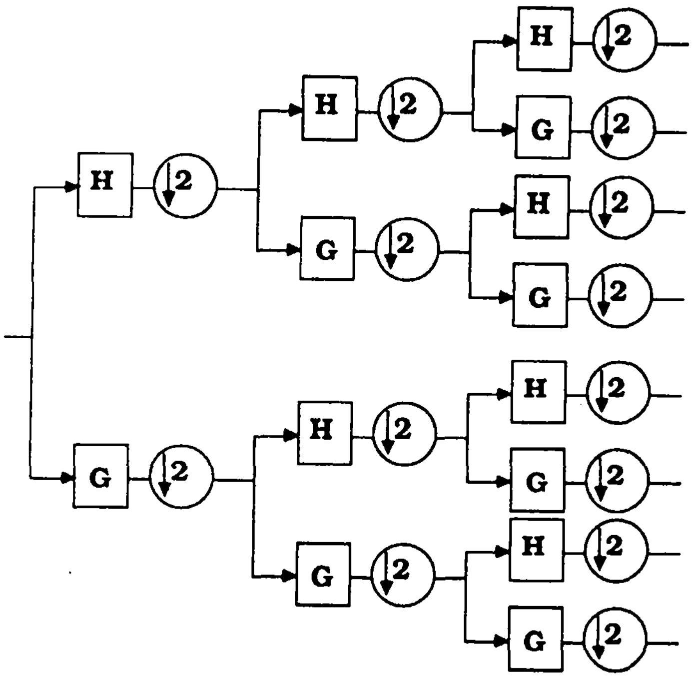


Figure 7: The Gabor tree

coarser scales, namely

$$\mathcal{V}_0 = \bigoplus_{w \in \{0,1\}^n} \mathrm{Im}\,(\mathcal{T}_{w_{-n}} \dots \mathcal{T}_{w_{-1}}) \tag{4.24}$$

where $w = w_{-n} \dots w_{-1}$, $w_{-i} \in \{0,1\}$, and

$$\mathcal{T}_0 = \mathcal{H}^*, \mathcal{T}_1 = \mathcal{G}^* \tag{4.25}$$

The name of the decomposition will be justified later on.

4.3 A wavelet coarse–scale decomposition

Using again (4.23), the alternative diagram of the figure 8 yields the decomposition

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{W}_{-1} \oplus \mathcal{W}_{-2} \oplus \dots \oplus \mathcal{W}_{-n} \oplus \mathcal{V}_{-n} \\ \mathcal{W}_{-m} &= \mathrm{Im}\left((\mathcal{H}^*)^{m-1}\mathcal{G}^*\right) \end{aligned} \tag{4.26}$$

In this case it is usual to take for $\{H, G\}$ a {low–pass,high–pass} pair.

REMARKS:

1. Draw the tree of the figure 7 until infinity. In this figure, it is cut according to vertical lines. In the figure 8 the cut was taken along a parallel to the upper branch. It should be clear from these comments that *many other patterns could be considered as well to yield different orthogonal decompositions.*

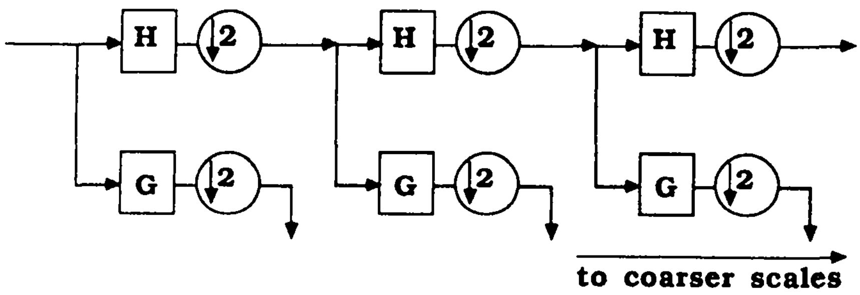


Figure 8: The wavelet tree

2. Assume we consider now a *non-unitary* QMF bank following section 3.1. Then all but the last formulae (4.22) remain valid. Hence (4.23) should be replaced by

$$\begin{aligned} \mathcal{V}_0 &\triangleq l^2(\mathbf{Z}) = \operatorname{Im}(\check{\mathcal{H}}) + \operatorname{Im}(\check{\mathcal{G}}) \\ &\triangleq \mathcal{V}_{-1} + \mathcal{W}_{-1} \end{aligned}$$

where $+$ refers here to complementary (but not necessary orthogonal) subspaces of $\mathcal{V}_0$. Hence both the gabor tree of the figure 7 and the wavelet tree of the figure 8 could be reinterpreted in this weaker sense, giving raise to non-orthogonal decompositions of $l^2(\mathbf{Z})$ into complementary subspaces of signals at coarser scales.

5 Introducing orthonormal wavelets

In this section, we shall introduce orthonormal wavelets via the asymptotic analysis of QMF banks.

5.1 Fine scale asymptotic behaviour of the orthonormal QMF bank, and multiresolution analysis of $L^2(\mathbf{R})$

Consider again the QMF analysis filter bank as shown in the figure 8. It yields a coarse-scale decomposition

$$\begin{aligned} l^2(\mathbf{Z}) &= \mathcal{V}_0 \\ &= \left(\bigoplus_{m=-1}^{-n} \mathcal{W}_m \right) \bigoplus \mathcal{V}_{-n} \end{aligned}$$

that can be also interpreted as a fine-scale decomposition

$$\begin{aligned} l^2(2^{-n}\mathbf{Z}) &= \mathcal{V}_n \\ &= \left(\bigoplus_{m=0}^{n-1} \mathcal{W}_m \right) \bigoplus \mathcal{V}_0 \end{aligned} \tag{5.27}$$

(just change the name of the input signal space). This suggests to consider the "infinite length" synthesis bank of the figure 9, i.e. to take the limit for fine scaling. The corresponding infinite

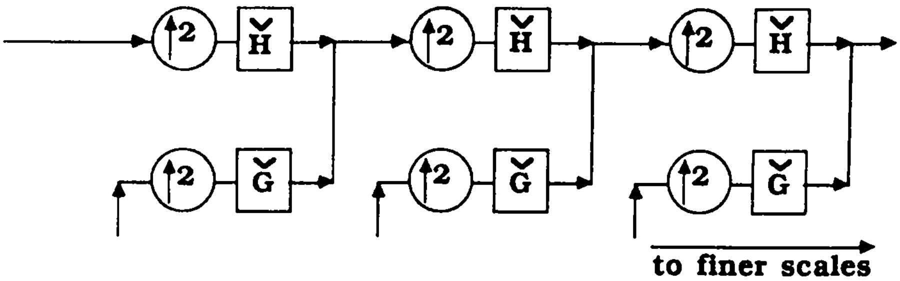


Figure 9: Infinite length synthesis filter bank

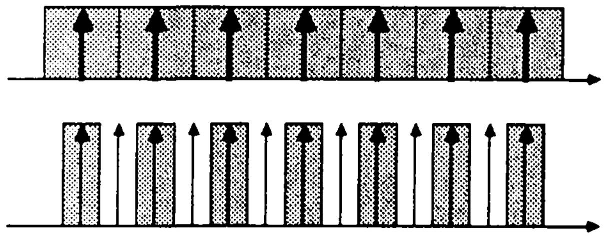


Figure 10: Upsampling in continuous time (note the scaling by a factor of $\sqrt{2}$).

block–diagram of the figure 9 exhibits a single infinite branch, namely the top one (all other branches cross this one at the first stage). Hence understanding this infinite behaviour amounts to study the top rightgoing branch, i.e. to ask for

$$\lim_{n\to\infty} (\mathcal{H}^*)^n . \delta_0 \; = \; ? \tag{5.28}$$

Recall that $\mathcal{H}$ is a (partial, i.e. non invertible) isometry. It should be clear that (5.28) is not the proper way to study the infinite behaviour of the synthesis filter bank: in this bank, the resolution is multiplied by 2 at each stage of the cascade, so that we expect to end up with an "infinite" resolution, something that is better represented with *continuous time index* rather than discrete one. Hence, what we shall do from the beginning is to imbed the discrete time index $l^2(2^{-n}\mathbf{Z})$–spaces into $L^2(\mathbf{R})$ by considering functions that are constant on the dyadic intervals of corresponding length. More specifically, denote by L^2_n the Hilbert space of the L^2–functions that are constant on the dyadic intervals of length 2^{-n}, and associate $l^2(2^{-n}\mathbf{Z})$ with L^2_n. The upsampling operator $\uparrow 2 \; : \; L^2_n \to L^2_{n+1}$ is carried on as shown in the figure 10. In this picture, each grey rectangle represent one element of the canonical orthonormal basis of L^2_n (top) and L^2_{n+1} (bottom) respectively. The scaling by $\sqrt{2}$ has been introduced to make $\uparrow 2$ a *unitary* injective operator. The translation of (5.28) in this new framework is presented now. Introduce

$$\chi_0 \triangleq \chi_{[-\frac{1}{2},+\frac{1}{2}[} \text{ (characteristic function)}$$

and consider the following recursion:

$$\begin{aligned} \chi_1 &\triangleq \overline{\mathcal{H}}^* \chi_0 \\ &= \sqrt{2} \sum_m h_m \chi_{[(m-\frac{1}{2})2^{-1},(m+\frac{1}{2})2^{-1}[} \end{aligned}$$

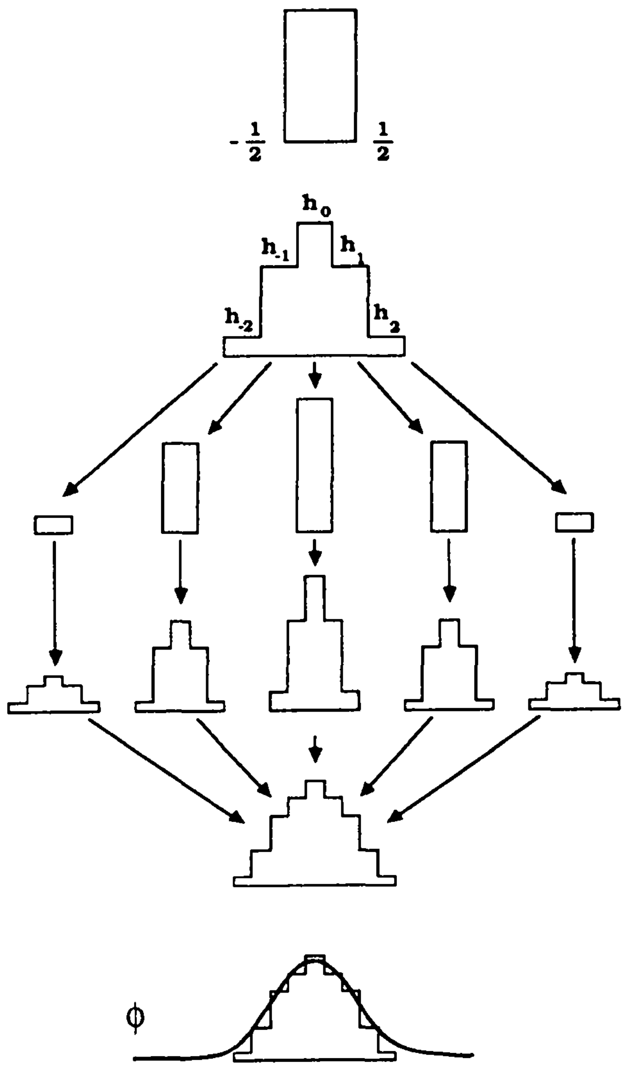


Figure 11: $\chi_0, \chi_1, \chi_2, \cdots, \phi$

$$\begin{aligned} \chi_2 &\triangleq \overline{\mathcal{H}}^* \chi_1 \\ &= 2 \sum_m h_m \left[\sum_p h_p \chi_{[(2m+p-\frac{1}{2})2^{-2},(2m+p+\frac{1}{2})2^{-2}[} \right] \end{aligned}$$

and so on. These two recursion stages are depicted in the figure 11, where the merging arrows refer to additive superposition. The corresponding recursion involves the following operator $\overline{\mathcal{H}}^*$:

$$\overline{\mathcal{H}}^* f(x) \triangleq \sqrt{2} \sum_m h_m f(2x - m) \tag{5.29}$$

Let us for the moment assume that we have proved the following :

$$\text{the limit } \phi \triangleq \lim_{n \to \infty} (\overline{\mathcal{H}}^*)^n \chi_0 \text{ exists in the } L^2\text{-sense,} \tag{5.30}$$

$$\{\tau^m \phi\}_{m \in \mathbf{Z}} \quad \text{orthonormal system in } L^2. \tag{5.31}$$

where $\tau f(x) = f(x+1)$ denotes the translation by 1. Then the following algebraic properties can be derived based on the properties of the orthonormal QMF synthesis bank:

Algebraic properties of the limit ϕ

1. the function ϕ satisfies the following fixpoint equation:

$$\phi(x) = \sqrt{2} \sum h_m \phi(2x - m) \text{ , i.e. } \phi = \overline{\mathcal{H}}^* \phi \tag{5.32}$$
$$\tag{5.33}$$

2. introducing

$$\psi(x) = \sqrt{2} \sum g_m \phi(2x - m) \tag{5.34}$$

we have

$$\begin{aligned} &\psi \perp \phi \text{ , and} \\ &\{\tau^m \psi\}_{m \in \mathbf{Z}} \text{ orthonormal system in } L^2 \end{aligned} \tag{5.35}$$

3. Introducing the following translation and dilation operators (both are isometries of L^2),

$$\begin{aligned} \tau f(x) &= f(x+1) \text{ (translation)} \\ \sigma f(x) &= \sqrt{2} f(2x) \text{ (dilation) , } \sqrt{2}\tau\sigma = \sigma\tau^2 \end{aligned}$$

the following orthogonal decompositions of L^2 into "successive scales" hold [9]:

$$L^2(\mathbf{R}) = \bigoplus_{n=-\infty}^{+\infty} \overline{\mathcal{W}}_n \tag{5.36}$$

$$\overline{\mathcal{V}}_n = \bigoplus_{m=-\infty}^{n-1} \overline{\mathcal{W}}_m \tag{5.37}$$

where

$$\begin{aligned} \overline{\mathcal{V}}_n &= \text{span}\{\sigma^n \tau^m \phi, m \in \mathbf{Z}\} \\ \overline{\mathcal{W}}_n &= \text{span}\{\sigma^n \tau^m \psi, m \in \mathbf{Z}\} \end{aligned}$$

Proof (sketch of). Point 1. is immediate. For point 2., we use the formula

$$\begin{aligned} \left[\tau^k \phi\right](x) &= \sqrt{2} \sum_m h_m \phi(2x + 2k - m) \\ &= \sqrt{2} \sum_m h_m \left[\tau^{2k-m} \phi\right](2x) \\ &= \sigma \left(\sum_m h_m \left[\tau^{2k-m} \phi\right] \right)(x) \end{aligned}$$

and the corresponding one for ψ with g_m instead of h_m. This can be rewritten as

$$\sigma^{-1} \tau^k \phi = \sum_m h_m \tau^{2k-m} \phi \tag{5.38}$$

$$\sigma^{-1} \tau^l \psi = \sum_m g_m \tau^{2l-m} \phi \tag{5.39}$$

Hence, denoting by $< .,. >$ the scalar product in L^2 and using the fact that the $\tau^m\phi$ form an orthonormal system in L^2, we may write

$$\begin{aligned} < \tau^k\phi, \tau^l\psi > &= < \sigma^{-1}\tau^k\phi, \sigma^{-1}\tau^l\psi > \\ \text{(using (5.38,5.39))} &= \mathcal{H}\mathcal{G}^*\delta_{k-l} = 0 \end{aligned}$$

due to the identities (4.22). Similarly, using again (4.22), we have

$$\begin{aligned} < \tau^k\psi, \tau^l\psi > &= < \sigma^{-1}\tau^k\psi, \sigma^{-1}\tau^l\psi > \\ \text{(using (5.39))} &= \mathcal{G}\mathcal{G}^*\delta_{k-l} = \delta_{k-l} \end{aligned}$$

which finishes to prove the point 2, and furthermore proves that the systems that span the spaces $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}}_n$ are orthonormal.

To prove (5.37) we remark that (5.38) implies $\overline{\mathcal{V}}_{-1} \subseteq \overline{\mathcal{V}}_0$. On the other hand, for $\overline{\mathcal{V}}_0 \ni f = \sum_k c_k\sigma^{-1}\tau^k\phi$, we have also thanks to (5.38) $f = \sum_k c'_k\tau^k\phi$ where $c' = \mathcal{H}c$. A similar result holds for $\overline{\mathcal{W}}_{-1}$ and $\mathcal{G}$ respectively. Thus the identity (4.23) carries on to yield

$$\overline{\mathcal{V}}_0 = \overline{\mathcal{V}}_{-1} \oplus \overline{\mathcal{W}}_{-1}$$

and (5.37) follows by induction from the corresponding properties of the orthonormal QMF bank. To derive (5.36) — i.e. to prove that our decomposition actually spans the whole L^2-space — requires more technical work however, and we refer the reader to [9, 6]. Note that this latter result states formally that adding finer scales is a valid approximation procedure.□

By the way, what we have got is a

multiresolution analysis of $L^2(\mathbf{R})$

in the sense of S. Mallat [14, 15, 16] as reported in [9] and the system $\{\sigma^n\tau^m\psi, m, n \in \mathbf{Z}\}$ we have so obtained is termed an *orthonormal wavelet basis.*

REMARK: when a non-unitary QMF bank is used, a more careful use of similar techniques allows to derive the *bi-orthogonal* wavelets due to Albert Cohen and Ingrid Daubechies, cf. remark 2 of section 4.3.

5.2 Existence and properties of the limit ϕ: proof of (5.30,5.31).

Here we switch from *algebra* to *analysis.*

Existence of ϕ: proof of (5.30). Introducing the Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

the formula

$$\phi = \lim_{n\to\infty}(\overline{\mathcal{H}^*})^n\chi_0$$

is rewritten as follows,

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}}\prod_{n=1}^{\infty} H(e^{i2^{-n}\omega})$$

and the limit is defined pointwise. This limit belongs to L^2 since, thanks to the QMF property, we know that the finite products $(\overline{\mathcal{H}^*})^n\chi_0$ are uniformly bounded in L^2.

Proof of (5.31). The following equivalences hold:

$$\{\tau^m\phi, m \in \mathbf{Z}\} \text{ orthonormal} \Leftrightarrow \sum_{m\in\mathbf{Z}} |\hat{\phi}(\omega + m2\pi)|^2 \equiv 1$$

Introduce

$$\begin{aligned}\Phi(\omega) &\triangleq \sum_{m\in\mathbf{Z}} |\hat{\phi}(\omega + m2\pi)|^2 \\ u(\omega) &\triangleq |H(e^{i\omega})|^2\end{aligned}$$

Then

$$\begin{aligned}\phi = \overline{\mathcal{H}}^*\phi &\Leftrightarrow \hat{\phi}(2\omega) = \sqrt{2}H(e^{i\omega})\hat{\phi}(\omega) \\ &\Rightarrow P_u\Phi = \Phi\end{aligned}$$

where the operator P_u is defined by

$$2P_u f(2\omega) = u(\omega)f(\omega) + u(\omega+\pi)f(\omega+\pi) \tag{5.40}$$

But, since the filter H satisfies the orthonormal QMF condition, we have

$$u(\omega) + u(\omega + \pi) = 2 \tag{5.41}$$

i.e P_u is the *transition probability of a Markov chain on the unit circle.* Then, knowing that Φ is P_u-invariant, we want to deduce that *it must be a constant.* Using the following equivalences

$$P_u\Phi = \Phi \;\Rightarrow\; \Phi \equiv 1$$

$$\Uparrow$$

continuous P_u-invariant functions must be constant

$$\Updownarrow$$

P_u ergodic

J-P Conze, A. Raugi (IRMAR Rennes) gave in [4] necessary and sufficient conditions on u for this Markov chain to be ergodic. Recall that ergodicity is a generic situation for Markov chains, so that QMF banks generally yield orthonormal wavelets. Similar results have been obtained in [2] by A. Cohen (CEREMADE, Paris) with different techniques.

Smoothness conditions are useful when the so obtained basis of L^2 is used for harmonic or functional analysis. The following theorem holds [9]

Theorem 1 (I. Daubechies) *Assume H is selected of the following form*

$$H(z) = \left[\frac{1}{2}(1+z^{-1})\right]^N \tilde{H}(z) \tag{5.42}$$

where, in addition to be selected for H to satisfy the orthonormal QMF property, the fiter $\tilde{H}$ satisfies the following conditions: .

$$\sum |\tilde{H}_n||n|^\epsilon < \infty, \epsilon > 0$$
$$\sup |\tilde{H}(e^{i\omega})| = B < 2^{N-1}$$

then ϕ satisfies

$$|\hat{\phi}(\omega)| \leq C(1 + |\omega|)^{-N+(\log B)/(\log 2)}$$

and the convergence is pointwise.

This last inequality guaranties that ϕ has its M–th derivative in L^2 for some M, but this index M is much smaller than N (in practice, for the wavelets obtained by thhe method of [9], one has approximately $M \simeq 0.2N$). Better results on the regularity of ϕ given the decay rate of H may be found in [11, 3, 5, 6].

NOTA: (5.42) and the QMF condition together imply

$$\begin{aligned} G(e^{i\omega}) &= G'(e^{i\omega}) = ... = G^{(N)}(e^{i\omega}) \\ &= 0 \text{ for } \omega = 0 \end{aligned}$$

or, equivalently,

$$\hat{\psi}(0) = ... = \hat{\psi}^{(N)}(0) = 0$$

and, finally, via inverse Fourier transform,

$$\int x^n \psi = 0, \quad n \leq N \tag{5.43}$$

Such vanishing moment conditions appear to be extremely useful for approximating functions, or integral operators when a $2D$–theory is considered.

5.3 Some additional comments.

Consider again the diagram of the figure 7, and draw it infinitely long. As we already indicated, several patterns may be used to cut this infinite tree diagram, among them we just showed 2, namely the wavelet analysis filter bank (obtained via a cut which is parallel to the top branch of the tree), and the "Gabor" tree (obtained via vertical cuts of the tree). Each pattern gives raise to a different orthogonal decomposition of the space $\mathcal{V}_0$. And the same method we have used to introduce the wavelets may be used as well to derive alternative orthonormal bases of L^2. In particular, the "Gabor" tree has been used by Beylkin, Coifman, Rokhlin, and Y. Meyer, to get a basis of oscillating L^2–functions they call "Gaborets". We shall give in the last section of this course some indications of how such infinite products may be studied; results along those lines are found in [11] [4].

6 Efficient approximations of L^2–functions

I. Daubechies proved that, with further algebraic conditions on H, the following conditions may be satisfied:

$$\exists x_o \in \mathbf{R} : \int \phi(x + x_o)x^m dx = 0 : 1 \leq m < M \tag{6.44}$$

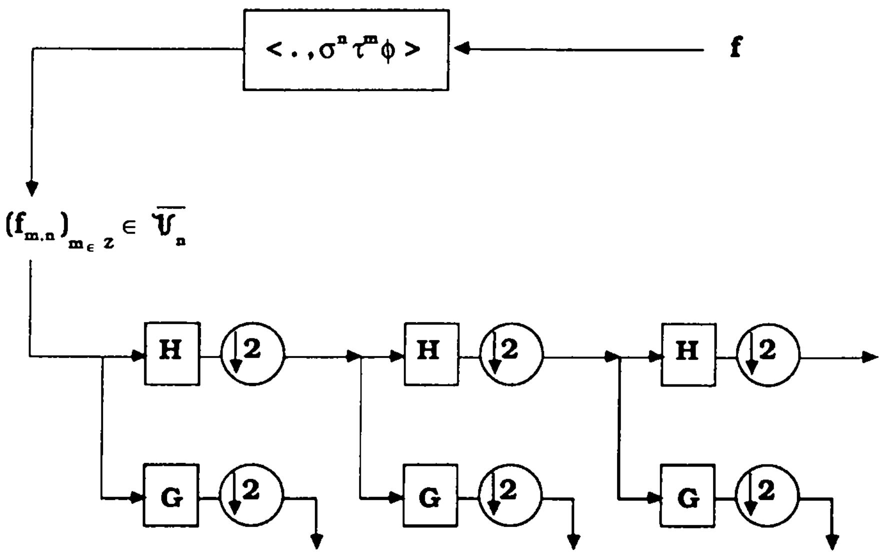


Figure 12: Expanding f.

Recall that, on the other hand,

$$\int \phi dx = 1$$

We shall use these vanishing moment conditions to compute efficient expansions of functions of the M-th order Sobolev space. For this purpose, we consider the diagram of the figure 12. This diagram describes a two–step approximation procedure:

1. project f onto the space $\overline{V}_n$ of L^2–functions at scale 2^{-n},

2. use the QMF analysis bank to further decompose the projection into an orthogonal expansion.

Since the second stage has already been investigated, only the following problem remains to be investigated: compute

$$< f, \sigma^n \tau^m \phi >$$

efficiently. This is explained next. Write the following equalities:

$$\begin{aligned} < f, \sigma^n \phi > &= 2^{\frac{n}{2}} \int f(x)\phi(2^n x)dx \\ &= 2^{-\frac{n}{2}} \int f(2^{-n}x)\phi(x)dx \\ &= 2^{-\frac{n}{2}} \int f(2^{-n}(x+x_o))\,\phi(x+x_o)dx \end{aligned}$$

A Taylor expansion of f around $2^{-n}x_o$ yields:

$$\begin{aligned} f\,(2^{-n}(x+x_o)) &= f\,(2^{-n}x_o) \\ &\quad +2^{-n}xf'\,(2^{-n}x_o) + \ldots \end{aligned}$$

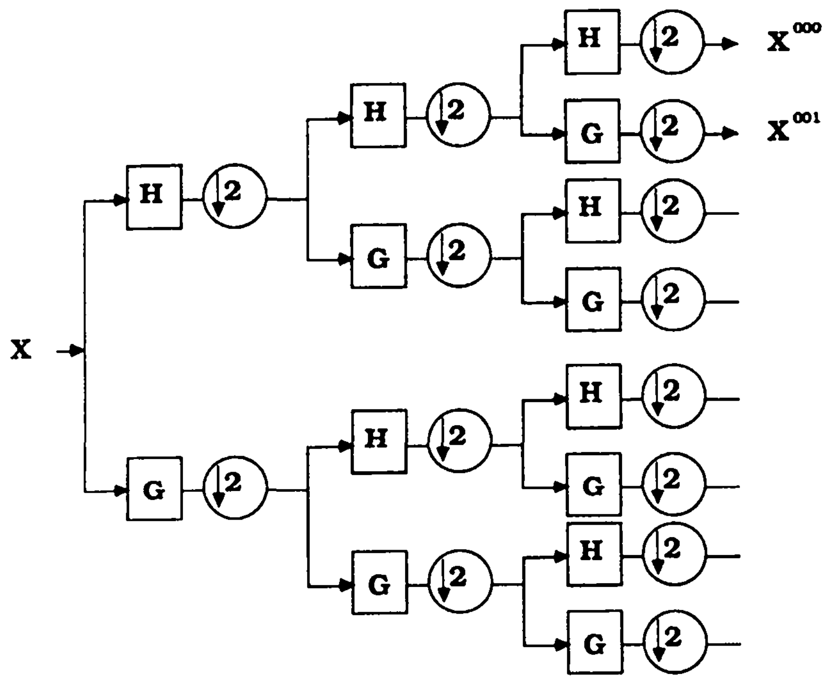


Figure 13: Signals through the "Gabor" tree

$$+2^{-(M-1)n}x^{M-1}f^{(M-1)}\left(2^{-n}x_o\right)$$
$$+O\left(\frac{2^{-Mn}}{M!}|x|^M\right)$$

so that, thanks to the vanishing moment conditions (6.44), we get the following *single point approximation !*

$$< f, \sigma^n\phi >= 2^{-\frac{n}{2}}f\left(2^{-n}x_o\right)+O\left(\frac{2^{-n(M+1/2)}}{M!}|x|^M\right)$$

$2D$-generalizations of this are useful to approximate integral Kernels involved in some integral equations (BCR algorithm, due to Beylkin, Coifman, and Rokhlin, see [1] [18]).

7 Application of orthonormal QMF banks to random signals

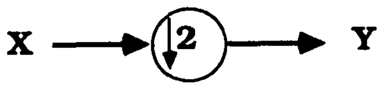
Consider again the "Gabor" tree of the figure 13. The following question has been considered by J-P Conze and A. Raugi [4]:

? What are the statistics of the signals on the leaves of the "Gabor" tree ?

More precisely, we want to investigate this question for very long trees.

7.1 Filtering and decimation of signals

Decimation of signals results in the following change of the spectral measure of the input signal:



Introduce the spectral measure of the process X:

$$\mathbf{E}X_0X_m = \int_0^{2\pi} e^{i\omega m}\mathcal{R}_X(d\omega)$$

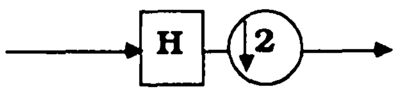
we have

$$\int f(\omega)\mathcal{R}_Y(d\omega) = \int f(2\omega)\mathcal{R}_X(d\omega) \tag{7.45}$$

If X has a spectral density, we have, denoting by $R_X(z)$ the spectrum of X:

$$R_Y(z) = \frac{1}{2}\left[R_X(z) + R_X(-z)\right] \tag{7.46}$$

Now, considering the case of filtering and decimation, we get



$$\int f(\omega)\mathcal{R}_Y(d\omega) = \int f(2\omega)|H(e^{i\omega})|^2\mathcal{R}_X(d\omega) \tag{7.47}$$

Again, if X has a spectrum we denote by R_X, we get:

$$\begin{aligned} R_Y(z) &= \frac{1}{2}\left[R_{HX}(z) + R_{HX}(-z)\right] \text{ where} \\ R_{HX}(z) &= H(z)R_X(z)H(z^{-1}) \end{aligned} \tag{7.48}$$

Case of a pure frequency

$$\begin{aligned} \mathcal{R}_X &= \delta_{\omega_o} + \delta_{-\omega_o} \\ \mathcal{R}_Y &= u(2\omega_o)\left[\delta_{2\omega_o} + \delta_{-2\omega_o}\right] \\ u(\omega) &\triangleq |H(e^{i\omega})|^2 \end{aligned} \tag{7.49}$$

Case of a spectral density

$$\begin{aligned} \mathcal{R}_X(d\omega) &= \mathcal{R}_X(e^{i\omega})d\omega \triangleq F(\omega)d\omega \\ \mathcal{R}_Y(d\omega) &= \mathcal{R}_Y(e^{i\omega})d\omega \triangleq P_uF(\omega)d\omega \end{aligned} \tag{7.50}$$

where P_u is the operator we already introduced:

$$2P_uf(\omega) = u(\omega)f(\omega) + u(\omega+\pi)f(\omega+\pi)$$

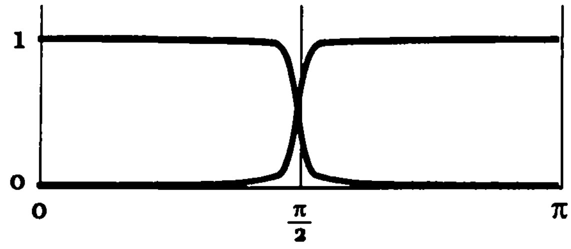


Figure 14: Low-pass / high-pass QMF bank.

7.2 Use of the "Gabor" tree

Consider again the figure 13, and introduce the following notations:

$$\begin{aligned} X &= (X_n), \mathcal{R}_X(d\omega) \\ \mathcal{T}_0 &= \mathcal{H}^*,\ \mathcal{T}_1 = \mathcal{G}^* \\ w &= w_{-n} \ldots w_{-1} \in \{0,1\}^n \\ \mathcal{T}_w &= \mathcal{T}_{w_n} \ldots \mathcal{T}_{w_1} \\ X^w &= \mathcal{T}_w X \end{aligned}$$

In other words, we encode the paths of the tree via a dyadic coding; when infinite paths are considered, the associated coding is the *dyadic expansion of the real number defined by this infinite path.* We are thus interested in the limit of the statistics of X^w when the length $|w|$ of the word w tends to infinity. Again, we consider two separate cases:

Case of a pure frequency. The following formula holds:

$$\mathcal{R}_{X^w} = \left[\prod_{m=1}^{|w|} u_{w_{-m}}(2^m \omega_o) \right] \left[\delta_{2^{|w|}\omega_o} + \delta_{-2^{|w|}\omega_o} \right] \tag{7.51}$$

where we used the notation

$$u_0(\omega) = |H(e^{i\omega})|^2, \quad u_1(\omega) = |G(e^{i\omega})|^2$$

Now, we make the following key remark:

$$2^m \omega_o \bmod 1 \ \text{ yields the dyadic expansion of } \omega_o \tag{7.52}$$

Now, assume $\{H, G\}$ is a {low-pass / high-pass} pair (cf figure 14): From the above condition and the formulae (7.52,7.51), we get the following asymptotic result, which holds for $|w|$ "infinitely" large:

$$\mathcal{R}_{X^w} \neq 0 \Leftrightarrow w_{-m} = 2^m \omega_o \bmod 1 \tag{7.53}$$

From this result, we derive the following

TEST: *the energy concentrates on the path encoding the dyadic expansion of* ω_o. By the way, what we got is a

$$\boxed{\textbf{multiscale isolation of pure frequencies}}$$

Case of a density $F(\omega)$ for X. Introduce the following notations:

$$\begin{aligned} P_i &\triangleq P_{u_i},\ i = 0, 1 \\ P_w &\triangleq P_{w_{-n}} \dots P_{w_{-1}} \\ F_w(\omega) &\triangleq P_w F(\omega) \text{ spectral density of } X^w \end{aligned}$$

What we want to investigate here is

$$\lim_{|w| \to \infty} F_w = ???$$

A new difficulty arises here, namely P_w is a *non homogeneous* Markov Transition kernel (recall that, thanks to QMF, we do have $P_w 1 = 1$). Such non homogeneous operators have been studied by I. Daubechies and J. Lagarias in [11] via tight bounds based on elementary linear operator algebra. On the other hand, Conze and Raugi [4] have imbedded this problem into that of the analysis of a single homogeneous Markov transition kernel: this is done by *randomizing the paths on the tree*, i.e. for each successive decimation, select H or G at random using a (fair) coin tossing. Hence w is randomized. Now consider the spectral density $F_w = P_w F$ of the process X^w: it is an L^2-function on the unit circle, and we have the following theorem:

Theorem 2 (Conze–Raugi) *For almost all infinite path w,*

$$P_w F \xrightarrow[n \to \infty]{L^2(\mathbf{T})} C_w$$

where C_w is a constant spectral density with a power depending on the path w, and the convergence holds in the L^2-sense on the unit circle $\mathbf{T}$. In other words, $X^w \to$ white noise with power C_w.

Case of several frequencies in coloured noise. Since $C_w \ll$ Dirac, an iterative multiscale procedure to isolate frequencies in coloured noise can be derived from the theorem above.

8 Conclusion

We have provided an account of those concepts of multiscale signal processing that are concerned with QMF techniques and orthonormal wavelet transforms. As opposed to most classical presentations of this subject, we started from discrete time signal processing and QMF banks and exlpoited their properties as far as possible to approach the construction of wavelets. Aside from providing a new insight on this topic, we think this presentation raised some additional questions, namely:

- How to exploit the suggestion of section 5.3 to generate other orthonormal decompositions of signal spaces and L^2-spaces?
- Is it possible to derive parametrizations of loss-less 2–port transfer functions in the style of section 3.3 to additionnally guaranty vanishing moment conditions such as provided by I. Daubechies' "explicit" examples? We suggest the so–called associated *chain scattering matrix* should be considered for factorization instead of the lossless transfer function itself.

ACKNOWLEDGEMENT: this short course is certainly an outcome of the "Analyse Multirésolution" working group at Rennes and surroundings[2], and it is worth to mention here Albert Cohen and Yves Meyer (both from CEREMADE, Paris), Patrick Flandrin (from Lyon), Loic Hervé, Albert Raugi, and also Ingrid Daubechies who was occasionnally invited to attend the group. But, first and mostly, the author is indebted to Jean-Pierre Conze, who performed an outstanding work in rephrazing the work of the classics on wavelet transforms to make it accessible to nonprofessionnal mathematicians, and furthermore provided exciting new uses and points of views. Jean-Pierre Conze is now preparing a detailed monography following the same philosophy as the present short course, we certainly recommand it for reading.

References

[1] G. BEYLKIN, R. COIFMAN, V. ROKHLIN, "Fast wavelet transforms", preprint Yale University, 1989.

[2] A. COHEN, "Ondelettes, analyses multirésolution, et filtres miroir en quadrature", preprint CEREMADE, Université de Paris IX Dauphine, 1989.

[3] A. COHEN, "Construction de bases d'ondelettes α–Hölderiennes", to appear in *Revista Matematica Iberoamericana*, 1991.

[4] J-P CONZE, A. RAUGI, "Fonctions harmoniques pour un opérateur de transition et applications," *Bull. Soc. Math. de France*, 118, 1990.

[5] J-P CONZE, "Sur la régularité des solutions d'une équation fonctionnelle", IRMAR res. rep., IRMAR, Université de Rennes I. 1990.

[6] J-P CONZE, course in preparation, IRMAR, Université de Rennes I.

[7] R.E. CROCHIERE, L.R. RABINER, *Multirate digital signal processing,* Englewood Cliffs, NJ: Prentice Hall, 1983.

[8] A. FETTWEISS, "Wave digital filters: theory and practice", *Proc. of the IEEE*, vol 74 No 2, 270-327, 1986.

[9] I. DAUBECHIES, "Orthonormal bases of compactly supported wavelets," *Communications on Pure and Applied Math.*, vol. 91, pp. 909–996, 1988.

[10] I. DAUBECHIES, "The wavelet transform, time-frequency localization and signal analysis," AT&T Bell Laboratories Report, to appear in *IEEE Trans. on IT.*

[11] I. DAUBECHIES, J. LAGARIAS, "Two–scale difference equations, II local regularity, infinite products of matrices, and fractals," preprint AT & T Bell Laboratories, 1989.

[12] D. ESTEBAN, C. GALAND, "Application of quadrature mirror filters to split–band voice coding schemes", Proc. of the ICASSP 1977, 191-195.

[13] W. HACKBUSCH AND U. TROTTENBERG, eds., *Multigrid Methods*, Springer-Verlag, New York, 1982.

[2]surroundings of Rennes include Rouen, Lyon, and Paris

[14] S.G. MALLAT, "A compact multiresolution representation: the wavelet model," Dept. of Computer and Info. Science—U. of Penn., MS-CIS-87-69, GRASP LAB 113, Aug. 1987.

[15] S.G. MALLAT, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. on PAMI,*, 11 No 7, July 1989.

[16] S.G. MALLAT, "Multiresolution approximation and wavelets orthonormal bases of $L^2(\mathbf{R})$," *Trans. Amer. Math. Soc.,* 315, Nol , 69-88, 1989.

[17] S. MCCORMICK, *Multigrid Methods*, Vol. 3 of the SIAM Frontiers Series, SIAM, Philadelphia, 1987.

[18] Y. MEYER, "Le calcul scientifique, les ondelettes et les filtres miroir en quadrature," preprint CEREMADE, Université de Paris IX Dauphine, 1990.

[19] Y. MEYER, *Wavelets and operators,* Proceedings of the Special year in modern Analysis, Urbana 1986/87, published by Cambridge University Press, 1989. See also Y. MEYER, *Ondelettes et Opérateurs,* Hermann, Paris, 1990.

[20] T.Q. NGUYEN, P.P. VAIDYANATHAN, "Structures for M-channel perfect reconstruction FIR QMF filter banks which yield linear phase analysis filters", *IEEE-ASSP 38 No 3,* 433-446, 1990.

[21] M.J. SMITH AND T.P. BARNWELL, "Exact reconstruction techniques for tree-structured subband coders," *IEEE Trans. on ASSP*, vol. 34, pp. 434–441, 1986.

[22] P.P. VAIDYANATHAN, "Quadrature mirror filter banks, M-band extensions and perfect-reconstruction techniques,"*IEEE ASSP Magazine 4 No 3,* 4-20, 1987.

[23] P.P. VAIDYANATHAN, "Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property", *IEEE ASSP-35,* 476-492, 1987.

PI 542 **A NEW APPROACH TO VISUAL SERVOING IN ROBOTICS**
Bernard ESPIAU, François CHAUMETTE, Patrick RIVES
Juillet 1990, 44 Pages.

PI 543 **SIMPLE DISTRIBUTED SOLUTIONS TO THE READERS-WRITERS PROBLEM**
Michel RAYNAL
Juillet 1990, 10 Pages.

PI 544 **IMPLEMENTATION AND EVALUATION OF DISTRIBUTED SYNCHRONIZATION ON A DISTRIBUTED MEMORY PARALLEL MACHINE**
André COUVERT, René PEDRONO, Michel RAYNAL
Juillet 1990, 14 Pages.

PI 545 **ESTIMATION OF NETWORK RELIABILITY ON A PARALLEL MACHINE BY MEANS OF A MONTE CARLO TECHNIQUE**
Mohamed EL KHADIRI, Raymond MARIE, Gerardo RUBINO
Août 1990, 20 Pages.

PI 546 **LIMIT THEOREMS FOR MIXING PROCESSES**
Bernard DELYON
Septembre 1990, 22 Pages.

PI 547 **PERFORMANCES DES COMMUNICATIONS SUR LE T-NODE**
Frédéric GUIDEC
Septembre 1990, 38 Pages.

PI 548 **LES PREDICATS COLLECTIFS : UN MOYEN D'EXPRESSION DU CONTROLE DU PARALLELISME *OU* EN PROLOG**
René QUINIOU, Laurent TRILLING
Septembre 1990, 34 Pages.

PI 549 **NORMALISATION SOUS HYPOTHESE D'ABSENCE DE LIEN APPLICATION AU CAS NOMINAL**
François DAUDE
Septembre 1990, 42 Pages.

PI 550 **MULTISCALE SIGNAL PROCESSING : FROM QMF TO WAVELETS**
Albert BENVENISTE
Septembre 1990, 28 Pages.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399